Difference Equation

Usually describes the evolution of certain phenomena over the course of time.

example: If a certain population has discrete generations, the size of $(n+1)$th generation

$x(n+1)$ is a function of the n th generation $x(n)$.

$$\Rightarrow \quad x(n+1) = f\big(x(n)\big)$$

autonomous (time invariant)

or, may be,

$$x(n+1) = g\big(n, x(n)\big)$$

non-autonomous (time-variant)

Linear 1st order Difference eqn

homogeneous 1st order

$$x(n+1) = a(n)\, x(n)$$

$$x(n_0) = x_0 \quad ; \quad n \geq n_0 \geq 0$$

The associated non-homogeneous,

$$x(n+1) = a(n)x(n) + g(n)$$

$$x(n_0) = x_0 \quad ; \quad n \geq n_0 \geq 0$$

In both eqn it is assumed that $a(n) \neq 0$ and $a(n), g(n)$ are real-valued functions defined for $n \geq 0$.

Linear 1st order Difference eqn

$$\boxed{x(n+1) = a(n)\, x(n)} \; ; \quad x(n_0) = x_0$$

$$n \geq n_0 \geq 0$$

$a(n) \neq 0$

We can obtain the solution of above equation through iteration

$$x(n+1) = a(n)\, x(n) \quad \Big| \quad \begin{array}{l} x(n_0) = x_0 \\ n \geq n_0 \geq 0 \end{array}$$

$$x(n_0+1) = a(n_0)\,x(n_0) = a(n_0)\,x_0$$

$$x(n_0+2) = a(n_0+1)\,x(n_0+1) = a(n_0+1)\,a(n_0)\,x_0$$

$$x(n_0+3) = a(n_0+2)\,x(n_0+2) = a(n_0+2)\,a(n_0+1)\,a(n_0)\,x_0$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$x(n_0+n) = a(n_0+n-1)\,x(n_0+n-1) = \cdots$$

$$\Rightarrow x(n) = \underline{a(n-1)} \cdot \underline{a(n-2)} \cdots\cdots a(n_0) \cdot \underline{x_0}$$

$$x(n) = \left[\prod_{i=n_0}^{n-1} a(i) \right] x_0$$

Solution for Linear homogeneous 1st order

Similarly, for the associated non-homogeneous eqn, we use iteration

$$x(n+1) = a(n)x(n) + g(n) \; ; \; x(n_0) = x_0$$

$$n \geq n_0 \geq 0$$

Using mathematical induction,

$$x(n) = \left[\prod_{i=n_0}^{n-1} a(i)\right] x_0 + \sum_{k=n_0}^{n-1} \left[\prod_{i=k+1}^{n-1} a(i)\right] g(k)$$

Example

Solve the equation

$$x(n+1) = (n+1)x(n) + 2^n (n+1)! \ ; \ x(0) = 1$$

Ans Using the formula, we can write

$$x(n) = \left[\prod_{i=n_0}^{n-1} a(i)\right] x_0 + \sum_{k=n_0}^{n-1} \left[\prod_{i=k+1}^{n-1} a(i)\right] g(k)$$

$$= \left[\prod_{i=0}^{n-1} (i+1)\right] (1) + \sum_{k=0}^{n-1} \left[\prod_{i=k+1}^{n-1} (i+1)\right] 2^k (k+1)!$$

$$= n! + \sum_{k=0}^{n-1} 2^k \cdot n! = [\text{P.T.O}]$$

$$= n! + n! \cdot \frac{(2^n - 1)}{2 - 1}$$

$$= 2^n \cdot n! \quad (\text{Ans})$$

H.W- Find the solution of the following

$$x(n+1) = a\,x(n) + g(n); \quad x(0) = x_0$$

$$\left[\text{Ans}: x(n) = a^n x_0 + \sum_{k=0}^{n-1} a^{n-k-1} g(k)\right]$$

H.W. Find the solution of the following

$$x(n+1) = a\,x(n) + b \;;\quad x(0) = x_0$$

Ans: $$x(n) = a^n x_0 + b \sum_{k=0}^{n-1} a^{n-k-1}$$

H.W Find a solution for the difference eqn.:

$$x(n+1) = 2 \cdot x(n) + 3^n \;;\quad x(1) = \frac{1}{2}$$

Ans: $3^n - 5 \cdot 2^{n-1}$

H. W. Find the solution for each of the following difference eqn.

(1) $x(n+1) - (n+1)\,x(n) = 0 \quad ; \; x(0) = C$

(2) $x(n+1) - 3^{n}\,x(n) = 0 \quad ; \; x(0) = C$

(3) $x(n+1) = x(n) + e^{n} \quad ; \; x(0) = C$

Ans: (1) $C \cdot n!$ (2) $C\, 3^{\frac{n(n-1)}{2}}$ (3) $C + \dfrac{e^{n}-1}{e-1}$

Ex:

(*) H.W.

A drug is administered once every 4 hours. Let $D(n)$ be the amount of the drug in the blood system at the nth interval. The body eliminates a certain fraction p of the drug during each time interval. If the amount administered is D_0, find $D(n)$ and $\lim_{n\to\infty} D(n)$

Ans: $\lim_{n\to\infty} D(n) = D_0/p$

Hint

$$D(n+1) = (1-p)\,D(n) + D_0$$

$$\Rightarrow D(n)$$

$$= \left[\prod_{i=0}^{n-1} (1-p)\right] D_0 + \sum_{k=0}^{n-1} \left[\prod_{i=k+1}^{n-1} (1-p)\right] D_0$$

$$= (1-p)^n D_0 + D_0 \sum_{k=0}^{n-1} (1-p)^{n-k-1}$$

= HW

$$\Rightarrow D(n) = \left(D_0 - \frac{D_0}{p}\right)(1-p)^n - \left(D_0/p\right)$$

Linear Difference Equation of Higher order

The general form of a kth order homogeneous linear difference eqn is given by

$$x(n+k) + p_1(n)\, x(n+k-1) + \cdots$$
$$\cdots + p_k(n)\, x(n) = 0$$

($= g(n)$ for non-homogeneous)

$p(n)$, $g(n)$ are real-valued functions defined for $n \geq n_0$ and $p_k(n) \neq 0$ for all $n \geq n_0$

Solution strategies:

Preliminaries (Terminologies)

1. **Fundamental set of solutions**

A set of k linearly independent soln of a kth order homogeneous linear differenc eqn is called fundamental set of soln

Linear dependence & Casoration

Using casoration $W(n)$, we can check the linear dependence of the soln

$$W(n) = \begin{vmatrix} x_1(n) & x_2(n) & \cdots & x_p(n) \\ x_1(n+1) & x_2(n+1) & \cdots & x_p(n+1) \\ \vdots & \vdots & & \vdots \\ x_1(n+r-1) & x_2(n+r-1) & \cdots & x_p(n+r-1) \end{vmatrix}$$

Fundamental set of solution & Casoration

The set of sol^n $x_1(n), x_2(n), \dots, x_k(n)$ of homogeneous eqn is a fundamental set, if for some $n_0 \in \mathbb{Z}^+$, the casoration $W(n_0) \neq 0$

Superposition principle

If $x_1(n), x_2(n), \dots, x_k(n)$ are solutions of the homogeneous eqn then $x(n) = x_1(n) + x_2(n) + \dots + x_k(n)$ is also a solution of homogeneous eqn

General solution:

Let $\{x_1(n), x_2(n), \cdots, x_k(n)\}$ be a fundamental set of soln of the homogeneous eqn.

Then the general soln is given by:

$$x(n) = \sum_{i=1}^{k} a_i x_i(n)$$

[a_i = arbitrary constant]